Picture this: from inscribed circle to Pythagorean proposition

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The Editor of *Mathematical Spectrum*, in one of his occasional capsules [10], recently confronted readers with two expressions for the radius r of the inscribed circle of a right triangle with legs a and b and hypotenuse c:

$$r = \frac{a+b-c}{2}, \qquad r = \frac{ab}{a+b+c}.$$

A natural response is to try setting these two expressions equal, whence a little algebra reveals that they are indeed equivalent, subject to the even more celebrated Pythagorean relation between a, b and c:

$$a^2 + b^2 = c^2. \tag{1}$$

This algebra is reversible to the extent that, given (1), each of the expressions for r implies the other. But it might seem rather artificial to go in these directions. However, as we shall *see*, the connection between the inscribed circle and the Pythagorean proposition is closer yet.

The two expressions for the radius of the inscribed circle of a right triangle have a long history, and already, some seventeen centuries ago, a Chinese mathematician, Liu Hui (220–280), gave a neat dissection argument that makes both transparent — an early instance of a *proof without words*. First of all, Liu Hui dismembers the right triangle as shown in Figure 1(i). Then, as in Figure 1(ii), he reassembles the pieces from four copies of the right triangle into a long rectangle whose sides we recognize to be the perimeter of the right triangle and the diameter d of the inscribed circle, that is, $a+b+c$ and $d = 2r$. From the alternative way we have placed these pieces in Figure 1(iii), or, more directly, from the dissection in Figure 1(i), we see that

$$a + b = c + d; \tag{2}$$

and, since the area of the four right triangles is conserved in the long rectangle,

$$2ab = (a+b+c)d. \tag{3}$$

Of course, these are just the equations from the editorial capsule [10] rewritten in terms of the diameter, rather than the radius.

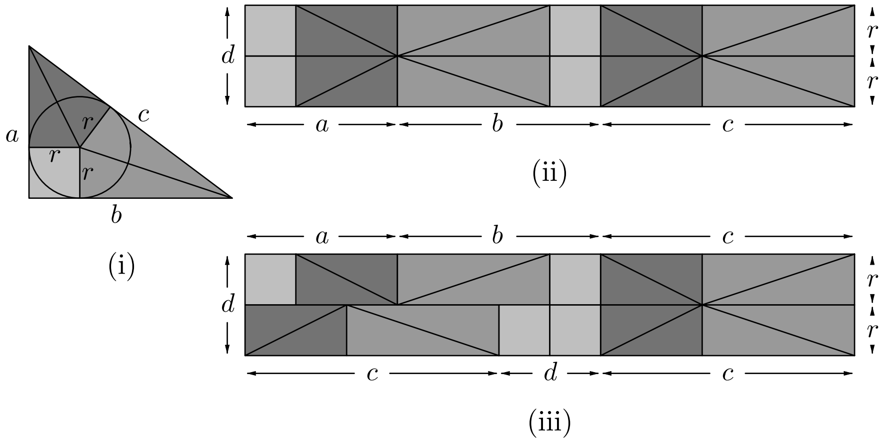


Figure 1: Liu Hui's dissection

Liu Hui's demonstration comes from an illustrated commentary on a then famous mathematical compilation, the *Jiu Zhang Suan Shu* (conventionally translated as *Nine Chapters on the Mathematical Arts*) from perhaps a century and a half earlier. Among the many problems contained in this work is, to give but one further instance, an early, exemplary version of Brahmagupta's problem of the broken bamboo, familiar to readers of *Mathematical Spectrum* from [9]. Unfortunately, in regard to the problem on the inscribed circle of a right triangle, so far from actually having a proof without words, the illustrative diagrams have disappeared, although what survives of the text indicates that they were coloured — for example, yellow for the little square of side r in Figure 1(i), crimson and indigo for the pairs of triangles. The reconstruction shown in Figure 1(ii) is standard, with the four small (yellow) squares grouped in pairs (sometimes with all four together), whereas in Figure 1(iii) they are placed so as to make (2) more apparent (*pace* [6, p. 104]).

But, now we have Figure 1(ii), we are free to play as we please with the set of twenty pieces that go to make up the rectangle, and to explore what other shapes can be obtained from them, much as in the game of *Tangrams*. Thus, we have in Figure 2 two further rearrangements of this set of pieces that bring the Pythagorean

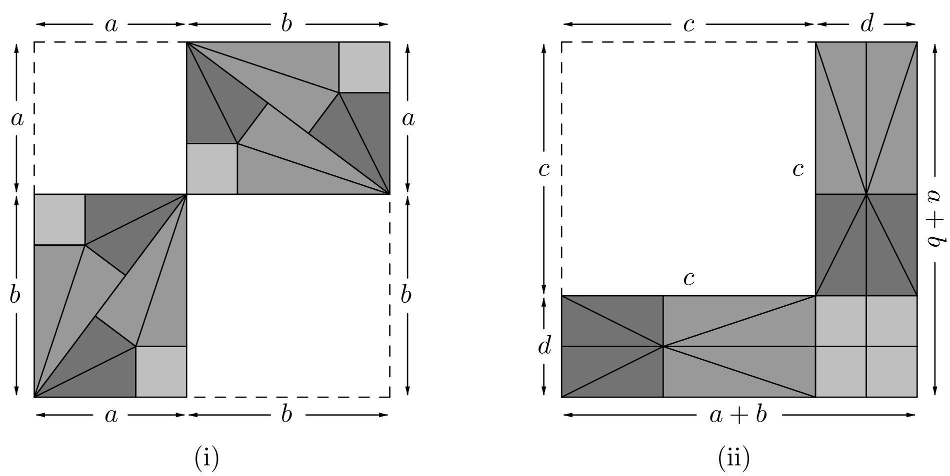
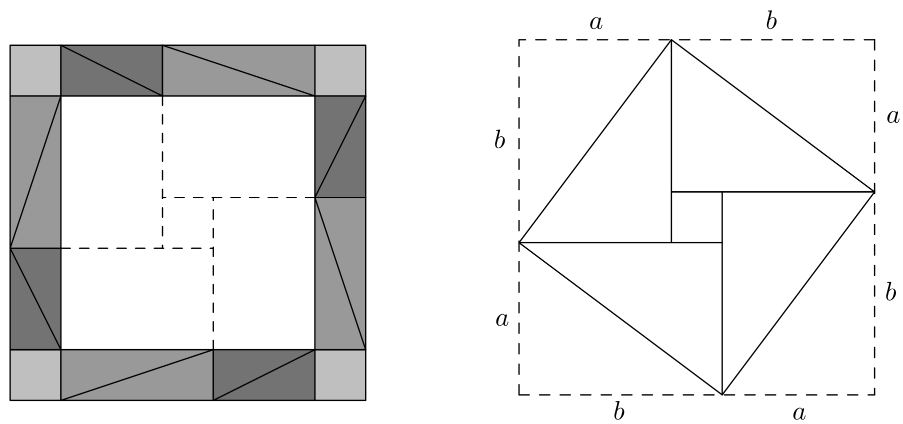


Figure 2: $a^2 + b^2 = c^2$

relation (1) into view, so to say, *in silhouette*. For, by (2), the containing rectangles in Figures 2(i) and 2(ii) have equal area. So, the complements of our set of twenty pieces in these rectangles have equal area, that is, the unshaded squares of sides a and b in Figure 2(i) and the unshaded square of side c in Figure 2(ii).

Now, making the rearrangements of the pieces shown in Figure 2 is only a matter of mathematical *play*, without suggestion that this has any *historical* basis. In fact, purely as part of this play, we observe that the selected rearrangements are such that the pieces of Figure 1(ii) can be slid with rotations into the positions in Figures 2(i) or 2(ii) *without turning them over*, as though in a board game. But still one might wonder whether it was within the *scope* of Liu Hui, knowing both that he favoured dissection arguments and that he seems to have accepted, in combination with them, demonstrations that turn on complementary figures? Liu Hui does discuss the Pythagorean relation (1), but the passage, as it has come down to us, is obscure, and possibly corrupt, so that it has been a matter of debate what he intended, beyond some kind of dissection. Several candidates are mentioned in the references below, each with its own champions.

In a further rearrangement of the twenty pieces in Figure 1(ii), we can form a frame inside a square of side $a+b$, so as to leave a square of side c aligned with it inside the frame, as in Figure 3(i). It is instructive to juxtapose this rearrangement of the pieces



(i) (ii)

Figure 3:

with a much more familiar diagram associated with the Pythagorean proposition, seen recently in *Mathematical Spectrum* in [5, p. 101, Fig. 4], and shown again here in Figure 3(ii) — a version of this diagram appeared in the logo for the International Congress of Mathematicians (ICM) held in Peking in 2002; and, indeed, there has been some presumption that it featured in a commentary from the same century as Liu Hui on another Chinese mathematical classic, the *Zhou Bi*. For Figures 3(i) and 3(ii) share the same *underlying* rotational symmetry, as suggested by the dotted lines in Figure 3(i); full rotational symmetry can be obtained in Figure 3(i) if we break the convention of not turning pieces over. Moreover, in view of Figure 1(i), each outer triangle in Figure 3(ii) has the same area as the corresponding L-shaped section of the frame, namely half the area of a rectangle with sides a and b. By sweeping the area of the four outer triangles into the frame, as it were, the inner square of side c is brought into alignment with the outer square of side $a + b$.

Again, this is only a mathematical observation, without any claim to historical foundation. But, curiously enough, a novel rendition of a passage from the *Zhou Bi* proposed in [3, p. 786] does speak of forming a ring of L-shaped trysquares, in conscious departure from the hitherto generally accepted translation (compare [1, p. 134, n. 37]).

The twenty pieces in Figure 1(ii) can be slid on their imagined board into two more configurations, as shown juxtaposed in Figure 4, where now the pieces are held in a

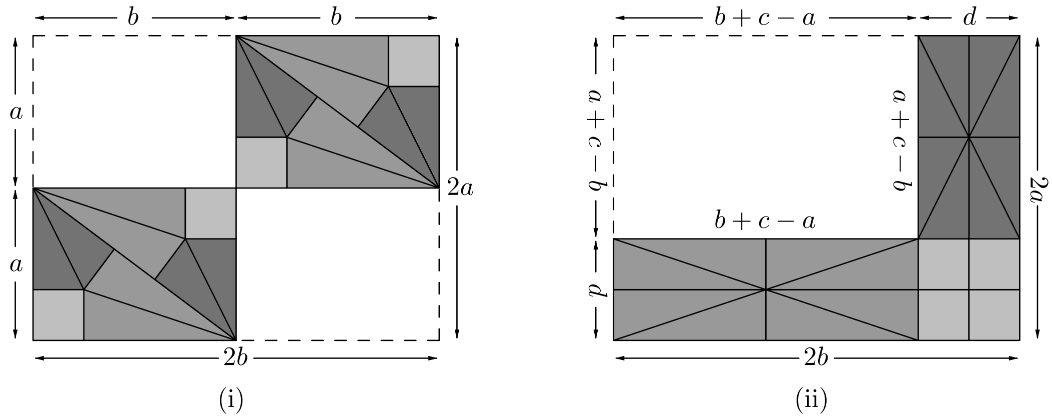


Figure 4: $2ab = (a + c - b)(b + c - a)$

common containing rectangle of sides $2a$ and $2b$. Reasoning as earlier with Figure 2, we see that the unshaded portion of this rectangle in Figure 4(i) has the same area as the unshaded portion in Figure 4(ii). Hence

$$2ab = (2a - d)(2b - d) = (a + c - b)(b + c - a),$$

where we have made use of (2) for the last equality. While this may seem no more than a minor variant on (3), both take on greater significance on recalling that, for $x = a, b, c$, the diameter d_x of the escribed circle of the right triangle with legs a and b and hypotenuse c touching the side x externally and the other two sides produced is given by

$$d_a = a + c - b, \quad d_b = b + c - a, \quad d_c = a + b + c.$$

Thus the same set of twenty tiles can be used to provide dissection demonstrations of four related results:

$$\begin{aligned} 2ab &= (a + b + c)d, & 2ab &= (a + b - c)d_c, \\ 2ab &= (b + c - a)d_a, & 2ab &= (a + c - b)d_b. \end{aligned} \tag{4}$$

On the other hand, it seems more difficult to establish directly by dissection that the shaded and unshaded portions of Figure 4(ii), like those of Figure 4(i), have equal area, or equivalently that

$$dd_c = d_a d_b,$$

although, as we see in Figures 2 and 4, both sides represent areas equal to $2ab$. (It may be of interest to note that if a triangle with sides a, b and c has inscribed and escribed circles of diameters d, d_a, d_b and d_c, then any of the conditions

$$dd_c = 2ab, \qquad d_a d_b = 2ab, \qquad dd_c = d_a d_b$$

is sufficient to ensure that the triangle is a right triangle with legs a and b and hypotenuse c.)

Once again, these are entirely mathematical observations. But it so happens that all four results in (4) appear in *Ce Yuan Hai Jing* (conventionally translated as *Sea Mirror of Circle Measurements*), a work by Li Ye (1192–1279) that was completed in 1248, so roughly a millennium after Liu Hui (see [8, pp. 43–149, esp. fig. 11.1]). Indeed, this book presents some 170 problems based on a single diagram in which, in effect, a circle is inscribed in and escribed to four similar right triangles. Commentators have often been struck by an apparent duality between problems in this collection. However, tackled by means of dissections of the sort used by Liu Hui, as with(4), Li Ye's set of problems loses some of this mystique: the problems are less difficult than commonly supposed; and it is possible to move between solutions to different problems quite easily. Liu Hui's commentary on the *Jiu Zhang Suan Shu* had been studied intensively, including for official examinations, in the intervening centuries. But the historical problem is the regard in which Li Ye and his contemporaries may have viewed these older dissection methods compared with the algebraic ones described in *Ce Yuan Hai Jing* and for which it is most noted.

The lack of documentation for Figures 2, 3(i) and 4 is not just a matter of the historical record, since they seem to be missing from more recent discussions too, both mathematical and pedagogical. If truly absent, it would seem strange that such a versatile set of shapes has attracted so little comment.

Mathematical Spectrum has already carried a general conspectus [5] of mathematics from Chinese antiquity. Some popular account of the work of Liu Hui is given in [11, 12, 4, 3] ([4] goes so far as to reproduce a version of Figure 1 in colours approximating Liu Hui's own choice). Two works of reference are [6, 8], while a more definitive account [2] of the *Jiu Zhang Suan Shu* has only recently appeared in French, supplementing a similar exercise [7] in English. But none of these works include mention of Figures 2, 3(i) or 4. A comment of Liu Hui regarding the way certain algorithms emerge from "the same transformation of one particular figure" has recently been taken up at length in [1]. That might sound somewhat akin to the various deductions made here merely by rearranging one set of twenty tiles in different ways. However, the figure on which [1] focuses is not one of these arrangements, being instead more closely related to Figure 3(ii).

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In the thirty years or so since graduation, Douglas Rogers has travelled widely with his sums. Consequently, he has become tolerably well used to picking up the pieces, and reassembling them.